ON LINK PATTERNS AND ALTERNATING SIGN MATRICES

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Abstract. We devise an algorithm to generate an alternating sign matrix(ASM) with the same blue and green link pattern on the circle. We also find a characterization of link patterns that are achieved by a unique ASM.

Contents

1. Introduction

Alternating sign matrices (ASM) are in one-to-one correspondence with *Fully Packed Loop (FPL) states* on the square ice lattice. A detailed introduction can be found in [2],[4]. In this model, each of the $2n$ blue(green) vertices is connected to one other blue(green) vertex by a path of blue(green) edges, or for short, vertex i is *linked* to vertex j.

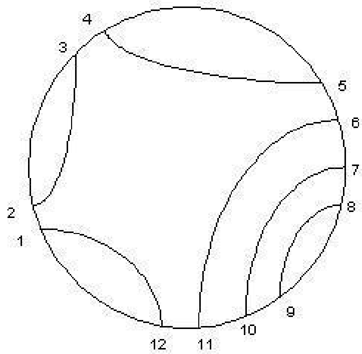
Definition 1.1. The *blue link pattern* π_B of an ASM is a set of n unordered pairs (r_k, s_k), where $1 \leq r_k < s_k \leq 2n$, such that the blue external vertex r_k is linked to the blue vertex s_k in the corresponding FPL model. The green link pattern π_G is defined similarly.

A blue or green link pattern of an ASM is equivalently realized as a *link pattern* of $2n$ vertices, labelled clockwise from 1 to $2n$, on the circle. In this case, each vertex is connected to exactly one other vertex by a chord, and the n chords do not intersect. It is an elementary fact that an odd-numbered vertex must pair up with an even-numbered vertex. Also, it is known there are totally $\frac{1}{n+1}\binom{2n}{n}$ link patterns of $2n$ vertices. An example of a link pattern on the circle is shown in figure 1.

Given a link pattern π_0, we are interested in knowing whether there always exists an ASM with both blue and green link patterns equal to π_0. We are able to show that this is true. To restate, our first main theorem is

Theorem 1.2. *For any given link pattern π_0 of $2n$ vertices on the circle, there exists an ASM of order n such that $\pi_B = \pi_0 = \pi_G$.*

FIGURE 1. A link pattern of 12 vertices



Two link patterns π_0 and π_1 are called *antipodal* if they differ by the 180 degree rotation. Inspired by theorem 1.2, we state and prove that

Theorem 1.3. *For any pair of antipodal link patterns π_0 and π_1, there exists an ASM of order n such that $\pi_B = \pi_0$ and $\pi_G = \pi_1$.*

We prove these two theorems by devising the respective algorithms to construct the desired ASMs. Let us fix any link pattern π_0 of $2n$ vertices. In section 2, we will present the *Skew Diagonal Algorithm(SDA)*, which generates an ASM(n) with $\pi_B = \pi_0 = \pi_G$. The proof of SDA is presented in section 3, and it consists of 2 big steps:

(1) we prove that SDA produces an ASM with $\pi_B = \pi_G$, in section 3.1;
(2) we show that $\pi_B = \pi_0$ in section 3.2.

In section 4, we construct, similar to SDA, the *Lattice Diagonal Algorithm(LDA)* to prove theorem 1.3. Since the proof of LDA is analogous to that of SDA, we will note the similarities and outline the proof in later of the section.

A direct application of the two algorithms is to characterize link patterns on the circle which are achieved by a unique ASM. To better describe these link patterns, we need a definition.

Definition 1.4. In a link pattern π_0 on the circle, an *adjacent link* is a link joining 2 adjacent external vertices.

Theorem 1.5. *For a link pattern π_0 of $2n$ vertices, there exists a unique ASM having π_0 as its blue link pattern if and only if π_0 has exactly 2 adjacent links.*

Zuber mentioned that Nguyen Anh-Minh proved the sufficient condition in [5]. In section 5, we prove that Zuber's condition is also necessary, and we give an independent proof of the sufficiency part.

In section 6 we will briefly discuss some possible generalization of our results.

2. ASMs with Identical Blue and Green Link Patterns

Before we start to present our first algorithm and the proof of theorem 1.2, we need some definitions.

Convention 1. From now on, when we discuss a FPL model, we will use *entries* to refer to *internal vertices* in the square ice lattice. Unless specified, *vertices* always refer to either *external vertices* of the square ice lattice or vertices on the circle.

2.1. **Some Definitions.**

Definition 2.1. A *tour* in an FPL state is a path of edges such that

(1) It starts from blue vertex 1;
(2) Blue and green edges in the path alternate;
(3) It only goes rightwards and upwards.

An entry in the tour is called a *tour entry*, and an edge in the tour is called a *tour edge*.

By abuse of notation, a tour in an ASM is a path of matrix entries that can be realized as a tour in the corresponding FPL state. Here the tour starts from the bottom left entry.

Definition 2.2. We define a sequence of *blue questions*, $B_\pi = \{b_e\}$, and a sequence of *green questions*, $G_\pi = \{g_f\}$, where each question is a true or false question on whether two blue(green) vertices *on the circle*, (i, j)(*unordered* pair), is in a link. The notation for the pth blue question is $b_p = ?(i, j)$, and for the qth green question it is $g_q = ?[k, l]$. Each question is either blue or green and not both.

First, all questions of the form $?(q, 2n+1-q)$, $q = 1, ..., n$ are blue questions. If $?(q, 2n+1-q)$ is true, then the next question is $?(q+1, 2n-q)$.

Suppose the blue question $?(q, 2n+1-q)$ is false. Then we start to alternate green and blue questions (green first) up to the next question of the form $?(p, 2n+1-p)$, $p > q$. All questions in between $?(q, 2n+1-q)$(included) and $?(p, 2n+1-p)$(excluded) are said to form a *cancellation cycle*, cycle for short. Thus, we form the sequence of all color questions, $Q_\pi = \{q_k\} = \{b_1, ..., b_{p_1}, g_1, b_{p_1+1}, g_2, ...\}$, by the order of appearance of each color question. It suffices to define color questions in a cycle.

Assume before the cycle starts, there are already $t = e + f$ color questions, with e blue questions and f green questions. The first green question in the cycle is $g_{f+1} = ?[q, q+1]$. Assume the kth green question in the cycle, is $g_{f+k} = ?[u_k, v_k]$. Moreover, $2 \le v_k < n$, and $k < n - q$. Then the $(k+1)$th green question (if applicable) is $g_{f+k+1} = ?[u_{k+1}, v_{k+1}]$ where $v_{k+1} = v_k + 1$ and

- if $?[u_k, v_k]$ is false, then $u_{k+1} = v_k$;
- otherwise, starting from u_k and going *counterclockwise*, u_{k+1} is the first vertex belonging to no links *confirmed* by *any* previous blue and green questions, i.e. the answers of all previous questions (if any) involving r_{k+1} are false.

The first blue question in the cycle is $b_{e+1} = ?(q, 2n+1-q)$. Suppose the kth blue question in the cycle, $b_{e+k} = ?(r_k, s_k)$, is asked, where $n+1 < s_k \le 2n$ and $k < n - q$. Then the $(k+1)$th blue question (if it is not of type $(p, 2n+1-p)$) is $b_{e+k+1} = ?(r_{k+1}, s_{k+1})$, where $s_{k+1} = s_k - 1$,and

- if $?(r_k, s_k)$ is false, then $r_{k+1} = s_k$; and
- otherwise, starting from r_k and going *clockwise*, r_{k+1} is the first vertex belonging to no links *confirmed* by previous blue and green questions.

The inductive definition from kth to $(k+1)$th blue question *in a cancellation cycle* is illustrated in figure 2.

The last possible green question is when $v_k = n$ and the last possible blue question is when $s_k = n + 1$.

The well-definedness of definition 2.2 is only challenged when after the last green question, we keep asking at least two blue questions not of the form $(p, 2n+1-p)$. But this is impossible, as follows from

Lemma 2.3. *After the last green question* $?[u, n]$ *appears, the next question is the last blue question* $?(r, n+1)$.

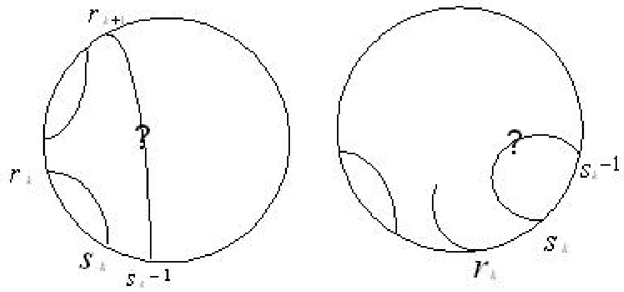


FIGURE 2. The $(k+1)$th blue question, $?(r_{k+1}, s_{k+1})$ in the cancellation cycle. Left: $?(r_k, s_k)$ is true; Right: $?(r_k, s_k)$ is false.

Proof. Assume that the last green question $?[u, n]$ belongs to the cancellation cycle starting from $?(p, 2n+1-p)$. Then the kth green question $?[u_k, p+k]$ is followed by the $(k+1)$th blue question $?(r_{k+1}, 2n+1-k-p)$. In particular, $?[u, n]$ is followed by $?(r_{n-p+1}, n+1)$, which by definition is the last blue question.

□

Moreover, a question is asked only once:

Lemma 2.4. *A color question never repeats itself in later of the sequence.*

Proof. The first question, $?(1, 2n)$ starts the first cancellation cycle. Assume the cycle is terminated by $?(p, 2n+1-p)$, where $p > 1$. The kth green question is $?[u_k, k+1]$ and the kth blue question is $?(r_k, 2n+1-k)$. Then all vertices involved in the previous $2k$ questions are in between the arc from $k+1$ counterclockwise to $2n+1-k$. But the $(k+1)$th blue question in the cycle is $?(r_{k+1}, 2n-k)$, so it is distinct from all previous questions. Also $r_{k+1} \neq k+2$, so it is also distinct from the $(k+1)$ green question. Similarly, the green question $?[u_{k+1}, k+2]$ is not repeated before.

Now at the end of this cycle, all vertices $p-1, ..., 1, 2n, ..., 2n+2-p$ have been confirmed their link status; otherwise, let such vertex be r, then either the green question at p is $?[r, p]$ or the blue question at $2n+1-p$ is $?(r, 2n+1-p)$, contradiction. In particular, the question $?(p, 2n+1-p)$ is not repeated before. Each subsequent blue question $?(s, t)$ has $2n+1-p \geq t \geq n+1$, and each subsequent green question $?[u, v]$ has $p \leq v \leq n$, so does not duplicate any previous question.

Now we can delete all vertices from the first cycle, relabel the remaining vertices clockwise with p being 1 and $2n+1-p$ being $2n-2p$, and start the above argument over. Therefore no question repeats itself afterwards.

□

Convention 2. From now on through section 4, when we write a matrix $A = [a_{i,j}]$, the ith row is counted from the *bottom* row, i.e. the $(n-i+1)$th matrix row in the usual sense.

Definition 2.5. In a square matrix of order n, a *skew diagonal(SD)* is a line of matrix entries parallel to the matrix diagonal. The kth skew diagonal, $k = 1, ..., 2n-1$, is the set of entries

- $k = n : a_{r,n-r+1}, r = 1, ..., n$ (the matrix diagonal)

- $k < n : a_{r,k-r+1}, r = 1, ..., k$
- $k > n : a_{k-n+r,n-r+1}, r = 1, ..., 2n - k$

Here we note that, for any particular tour in an ASM, there is at most one tour entry in each skew diagonal, since the tour only goes upwards and rightwards.

Definition 2.6. The *positive direction* of a skew diagonal is going from the bottom right entry towards the top left entry.

Definition 2.7. In an incomplete matrix, if an entry $a_{r,s}$ with $r \leq s$ is determined, then *Skew Diagonal Filling(SDF)* at $a_{r,s}$ is to set $a_{p,q}$=$a_{r,s}$ if

(1) $a_{p,q}$ is on the SD of $a_{r,s}$, and
(2) $p \leq q$ and $p > r$.

Definition 2.8. In a square matrix of order n, the *standard lattice diagonal(SLD)* is the line of entries $a_{r,r}$, $r = 1, ..., n$.

Definition 2.9. In an incomplete matrix, if an entry $a_{r,s}$ is determined, then the *Reflection Construction(RC)* at $a_{r,s}$ is to set $a_{s,r} = a_{r,s}$, i.e. the image of $a_{r,s}$ under reflection by the standard lattice diagonal.

2.2. **The Skew Diagonal Algorithm(SDA).** Here we present the *Skew Diagonal Algorithm(SDA)* that generates an ASM with $\pi_B = \pi_0 = \pi_G$, given a link pattern π_0 on the circle.

The basic mechanism of SDA is as follows. First, we record the question sequence $Q_\pi = \{q_k\}$ in definition 2.2 on π_0, and also the answers to all the questions. Then starting from an empty matrix, we construct the tour and based on the tour, construct the ASM. Once a matrix entry is determined, we do RC at this entry, and an entry, once determined, will never change. For a tour entry $a_{p,q}$, if it is the kth tour entry below or on the SLD, then we check the answer of kth color question, determine this entry, and decide what the next tour entry is. If $a_{p,q}$ is above the SLD, we always proceed 1 entry rightwards to get back to below or on the SLD. After a *tour* entry is determined, we do SDF.

If an ASM is invariant under reflection about the standard lattice diagonal, then by reflecting the corresponding FPL state and switching the colors of all edges, each blue link (i, j) is identified with the green link with same vertices, so $\pi_B = \pi_G$. Therefore we stipulate

Convention 3. Throughout the algorithm, whenever an entry $a_{p,q}$ is determined, do RC at $a_{p,q}$ as in definition 2.9.

We first start with the empty $n \times n$ matrix. Here we already have recorded $Q_\pi = \{q_k\}$ and the answers of each q_k.

Step 1. The first tour entry is $a_{1,1}$. Here check the answer of the first question in Q_π. If the answer is true, set $a_{1,1} = 1$, do SDF at this entry, set $a_{1,j} = 0$ if $j > 1$, and proceed to step 4. Otherwise, set $a_{1,1} = 0$, the next tour entry is $a_{1,2}$, and proceed to step 2.

Step 2. Suppose we are at the tour entry $a_{1,m}$, $m \geq 2$. Check the answer of the mth color question. More explicitly, check the answer of the kth green question if $m = 2k$, and check the $(k + 1)$th blue question if $m = 2k + 1$, for $k \geq 1$. If the answer is true, then set $a_{1,m} = 1$, do SDF at this entry, set $a_{1,j} = 0$ if $j > m$, and proceed to step 4. Otherwise, set $a_{1,m} = 0$, do SDF at this entry, the next tour entry is $a_{1,m+1}$, and go back to the beginning of this step.

Step 1 and 2 together determine the first row.

From step 3 through 7, assume at the tour entry $a_{r-1,m_{r-1}}$, we already check the answers of first $(d - 1)$ questions. So the question checked at $a_{r-1,m_{r-1}}$ is the dth question where $d = e + f$, with e checked blue questions and f checked green questions. Assume the dth question is TRUE and the entry $a_{r-1,m_{r-1}}$ is determined, the eth blue question is $?(r_e, s_e)$ and the fth green question (if any) is $?[u_f, v_f]$. These steps are to determine rth row and then set up new induction from rth to $(r+1)$th row.

Step 3. Do SDF at $a_{r-1,m_{r-1}}$. Set $a_{r-1,t} = 0$ if $t > m_{r-1}$. The $(r-1)$th row is then determined.

Step 4. If $m_{r-1} = r - 1$, then the next two tour entries are $a_{r,r-1}$ and $a_{r,r}$, and proceed to step 5. Otherwise the next tour entry is $a_{r,m_{r-1}}$ and proceed to step 6.

Actually, $a_{r,r-1} = a_{r-1,r} = 0$ by RC.

Step 5. Now at $a_{r,r}$, check the answer of $(d+1)$th color question (in Q_π), which is actually $?(r, 2n+1-r)$. If the answer is true, then set $a_{r,r} = 1$, $m_r = r$, and go back to step 3, where $r-1$ becomes r. The rth row is then determined. If the answer is false, then set $a_{r,r} = 0$, do SDF at this entry, the next tour entry is $a_{r,r+1}$, and proceed to step 7.

Step 6. At $a_{r,m_{r-1}}$, check the answer of $(d+1)$th color question, which is actually the $(f+1)$th green question if dth question is blue, and the $(e+1)$th blue question if dth question is green. If it is true, then set $a_{r,m_{r-1}} = 0$, $m_r = m_{r-1}$, do SDF at this entry, and go back to step 3 where $r-1$ becomes r. If it is false, set $a_{r,m_{r-1}} = -1$, do SDF at this entry, and the next tour entry is $a_{r,m_{r-1}+1}$, then proceed to step 7.

Step 7. At $a_{r,m}$, $m_{r-1} < m \leq n$, check the answer of $(d+m-m_{r-1}+1)$th question in Q_π. If the answer is true, then set $a_{r,m} = 1$ and $m_r = m$, do SDF at $a_{r,m}$, and go back to step 3 where $r-1$ becomes r. Otherwise, the answer is false, set $a_{r,m} = 0$, do SDF, and the next tour entry is $a_{r,m+1}$, and go back to the beginning of this step.

The algorithm terminates if all the questions in Q_π have been checked. However, it is not entirely clear the SDA can even complete a matrix, or the matrix is an ASM. We will devote the section 3.1 to prove this. We illustrate the SDA with the following example:

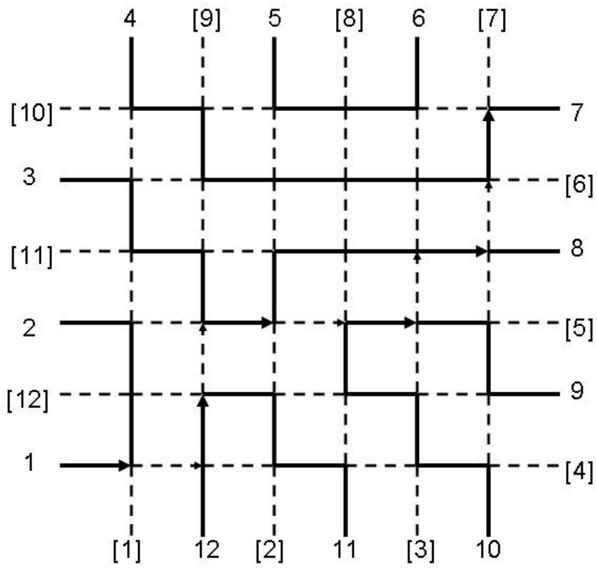
Example 2.10. Let $\pi_0 = \{(1,2); (11,12); (9,10); (3,8); (4,7); (5,6)\}$. We list the sequential color questions and their answers in the following table:

Question	Answer	Tour entry	Remark
Blue $?(1,12)$	False	$a_{1,1} = 0$	
Green $?[1,2]$	True	$a_{1,2} = 1$	Set $a_{1,m} = 0$ for $m > 2$
Blue $?(12,11)$	True	$a_{2,2} = 1, a_{3,2} = 0$	Set $a_{2,m} = 0$ for $m > 2$
Blue $?(3,10)$	False	$a_{3,3} = 0$	
Green $?[3,4]$	False	$a_{3,4} = 0$	
Blue $?(10,9)$	True	$a_{3,5} = 1$	Set $a_{3,m} = 0$ for $m > 5$
Green $?[4,5]$	False	$a_{4,5} = -1$	
Blue $?(3,8)$	True	$a_{4,6} = 1$	π_0 is determined
Green $?[5,6]$	True	$a_{5,6} = 0$	
Blue $?(4,7)$	True	$a_{6,6} = 0$	

Now we apply the SDA and obtain

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The FPL state of this ASM is shown in the figure below. The edges with arrows constitute the tour constructed by the SDA as realized in the FPL state.



3. Proof of the SDA

Our main theorem in this section, which directly implies Theorem 1.2, is

Theorem 3.1. *Given a link pattern π_0, the SDA generates an ASM with $\pi_B = \pi_0 = \pi_G$.*

First, we will prove that the SDA always completes the matrix and the matrix is an ASM with $\pi_B = \pi_G$. This is intimately related to how the answers of blue and green questions determine π_0. This part constitutes section 3.1.

Second, we establish an exact translation of the color questions to the construction of the FPL state of our ASM, so that $\pi_B = \pi_0$. This is treated in section 3.2.

Throughout this section, we will fix a link pattern π_0 of $2n$ vertices, and the tour always refers to the one concerned in the SDA. Whenever convenient, we will use *SD* as an abbrievation of skew diagonal.

3.1. **Determination of a Link Pattern.** We first define

Definition 3.2. Consider the sequence of vertices $r_1 = 2n+1-\lfloor\frac{n}{2}\rfloor$,...,$r_{\lfloor\frac{n}{2}\rfloor} = 2n$, $r_{\lfloor\frac{n}{2}\rfloor+1} = 1$,...,$r_n = \lceil\frac{n}{2}\rceil$. If a link (u, v) on the circle has at least a vertex in the above sequence, then we call it a *bottom adjacent link*.

On the square ice lattice, the n vertices in above definition corresponds to the n external vertices in the bottom, but undistinguished in their colors.

Lemma 3.3. *There exists a blue or green bottom adjacent link.*

Proof. Suppose there is no bottom adjacent link, then any two of the vertices $r_1, ..., r_n$ cannot be linked. Then the remaining n vertices are $s_k = r_1 - k$, and π_0 must be (r_1, s_1),...,(r_n, s_n). But (r_1, s_1) is a bottom adjacent link. □

Actually, the above lemma corresponds to the situation that the answers of the first $(n-1)$ color questions are all false. These answers uniquely determine π_0 in the above proof. The general condition to determine π_0 through color questions is

Proposition 3.4. *We ask blue and green questions as indicated in definition 2.2. Then:*

(1) *There exists a color question in the sequence Q_π, such that when this question is answered, we have obtained totally p true answers, and there are exactly $n-p$ consulted (i.e. involved in a previous question) but unconfirmed (i.e. involved in a previous true question) vertices.*
(2) *Assume (1) happens, then π_0 is uniquely determined by the answers of all previous questions;*
(3) *After the determination point in (1), there are $n-p$ subsequent color questions and all of them are true.*

Proof. Do induction on the total number p of true answers before (1) happens.

Let $p = 0$. Starting from the first question, we always get false answers. The first question, $?(1, 2n)$, produces 2 consulted but unconfirmed vertices. Since then, any two consecutive questions share a vertex. Therefore, after the $(n-1)$th question, we will obtain n consulted but unconfirmed vertices. This proves (1). For (2), by lemma 3.3, the link pattern is determined as $\pi^0 = \{(r_1, s_1), ..., (r_n, s_n)\}$ in the proof of lemma 3.3. For (3), the nth question and $(n+1)$th questions are $?(r_1, s_1)$ and $?[s_n, r_n]$ (or reverse), etc. Each subsequent question matches a pair in π^0, so is true. On the other hand, in this case there are totally $2n-1$ color questions, so there are n subsequent questions after the determination point in (1).

Assume the case $p = l - 1$ is true. For $p = l$, suppose the last true answer is of blue question $?(u_0, v_0)$. Then all vertices clockwise from v_0 to u_0 are confirmed. We delete the vertices u_0 and v_0 from the circle, and relabel the remaining $2n-2$ vertices. The number of consulted but unconfirmed vertices remain the same, while there is one less true answers. Since $n-p = (n-1)-(p-1)$, by induction hypothesis the case $p = l$ holds. □

Lemma 3.5. *When the tour proceeds to $a_{p,q}$ during the SDA, there are $q-p+1$ consulted but unconfirmed vertices if $q > p$, or if $q = p$ and the previous tour entry is $a_{p-1,p}$; otherwise, there is none at the tour entry $a_{p,p}$.*

Proof. Induction on the number p in $a_{p,q}$. For $p = 1$, the latter statement is trivial, and if the first question is false, there are 2 such vertices (namely, 1 and $2n$). From then, note that if the question at $a_{1,q}$ is false, then we obtain 1 new such vertex, since this question shares a vertex with the previous question. Therefore the case $p = 1$ is justified.

Assume the case $p = s$ and the last tour entry in sth row is a_{s,q_0}. We delete vertices confirmed in the first $p-2$ true answers. By induction hypothesis, if $q_0 = s$, there is either none or only 1 consulted but unconfirmed vertex in the new set of vertices. Then at $a_{s+1,s+1}$, there is no such vertex. If $q_0 > s$, then the last question introduces only one new vertex, thus there is one less unconfirmed vertex at a_{s+1,q_0}. The same argument in the last paragraph can be used once we delete the two vertices in the last true answer. This justifies the case $p = s+1$. □

An important consequence of proposition 3.4 and lemma 3.5 is

Lemma 3.6. *The tour always ends at* $a_{n,n}$*. Moreover,* $a_{p,n} = 1$ *where* $a_{p,n}$ *is the first tour entry in* n*th column, and* $a_{r,n} = 0$ *if* $r > p$*.*

Proof. Let the determination point in proposition 3.4 occur when we have obtained $p-1$ true answers. Then let the first tour entry in the pth row be $a_{p,q}$, where $n > q > p$. By lemma 3.5, at $a_{p,q}$ we only have $q-p+1$ consulted but unconfirmed vertices. Since we have to go through all the questions, by proposition 3.4, $n-q$ more such vertices are required. This is equivalent to get $n-q$ more false answers, since two consecutive questions, with the first being false, share a vertex. By step 7 the tour reaches $a_{p,n}$. This argument is the same for the case $q = p$ and the last tour entry is $a_{p-1,p}$.

In the case $q = p < n$ with the last tour entry being $a_{p,p-1}$, we need $n-p+1$ vertices, which are covered by the subsequent $n-p$ false answers of color questions. However, if $p = n$, then we are left with the last color question $?(n, n+1)$, which must be true. In this case, we arrive at the last tour entry $a_{n,n} = 1$.

Now by proposition 3.4, all the subsequent $n-p$ questions are true, so by step 7 of the SDA, $a_{p,n} = 1$, and $a_{r,n} = 0$ if $r > p$.

□

In particular, the SDA always completes a matrix in the end. Now we note that all entries below the tour are 0. Thus we reach our first goal by the following proposition:

Proposition 3.7. *The SDA generates an ASM with* $\pi_B = \pi_G$*.*

Proof. $\pi_B = \pi_G$ is obvious once we prove the matrix is an ASM, since by construction it is invariant under reflection about the standard lattice diagonal.

Consider the kth column for $k = 1, ..., n$. From the bottom, there exists a tour entry in this column (lemma 3.6), and we turn rightwards, find the 1 at the rightmost tour entry in this row, and pass it back to kth column. This is the first nonzero entry from the bottom. Now starting from the rightmost tour entry in kth row, we go backwards in the tour, find the first nonzero entry, and pass it back to kth column. This nonzero tour entry is 1 and passes back as the first nonzero entry from the top.

Now we prove 1 and -1 alternate. If there is a -1, pass it down to the tour, go rightwards, find the 1 at the rightmost tour entry in this row, and pass it back. This 1 is the next nonzero entry. If there is a 1 and still there is a nonzero entry to follow, then again we pass it down to the tour. Starting from this tour entry, once the tour arrives at the SLD, all the succeeding tour entries cannot be passed to the kth column, and thus the entries above the 1 are all 0. So the next nonzero tour entry to pass on to kth column is below the SLD, therefore it must be -1.

By reflection, every row satisfies the same conditions.

□

Note that to this stage, we don't know whether $\pi_B = \pi_0$. The next section is to prove this.

3.2. **FPL Configurations on a Skew Diagonal.** First we would like to realize the tour constructed by SDA in the FPL model, as follows:

Lemma 3.8. *The edges in the tour are sequentially constructed by the following rules:*

1. *Blue and green colors alternate;*
2. *If a top tour edge appears at* $a_{r,r}$*, then the next tour edge is on RHS of* $a_{r+1,r}$ *(actually it is blue).*

(3) *If the question checked at $a_{p,q}$ is true, then fill the next edge on top of the entry; otherwise, fill it on the RHS.*

Proof. Do induction on the cardinal k of the skew diagonal of a tour entry. The tour starts with a blue edge connected to blue vertex 1 . For $k = 1$, either $a_{1,1} = 1$ (true answer) where the next tour edge is green on top of $a_{1,1}$, or $a_{1,1} = 0$ (false answer) where the tour edge is green on the RHS.

Now suppose it is true for $k = l$. Let the tour entry on $(l + 1)$th SD be $a_{p,q}$. First let $q = p - 1$. Then $a_{m,p-1} = 0$ for all $m > p - 1$ by reflection of $(p - 1)$th row, and the sum of all $a_{r,p-1}, r = 1, ..., p - 1$, is 1. Therefore, from the bottom vertical edge of $(p - 1)$th column in the square ice lattice up to the bottom edge of $a_{p,p-1}$, the color changes evenly many times if $p - 1$ is odd, and oddly many times if $p - 1$ is even. So the bottom edge of $a_{p,p-1}$ is always green. Therefore the next tour edge is blue on RHS of $a_{p,p-1}$.

Then let $q \geq p$. WLOG, assume the preceding tour edge of $a_{p,q}$ is blue, otherwise reverse all the colors below. If the answer at $a_{p,q}$ is true, we choose the top edge of $a_{p,q}$ to go to the next tour entry $a_{p+1,q}$ by step 4. This edge is green, since either $a_{p,q} = 1$ and the preceding edge is on LHS, or $a_{p,q} = 0$ and the preceding edge is at the bottom. Otherwise the answer is false, and we would choose the RHS edge and move rightwards by step 6 and 7. This edge is green both when $a_{p,q} = -1$ (bottom preceding edge) and when $a_{p,q} = 0$ (LHS preceding edge).

□

Proposition 3.9. *On the SD of a tour entry $a_{p,q}$ on kth SD:*

(1) *The blue paths that cross the kth SD at an entry between $a_{p,q}$ and $a_{q,p}$ are disjoint from each other;*
(2) *The entries below $a_{p,q}$ belong to a single blue path.*

In particular, if $a_{r,n}$=1, then the entries on any SD on top of $a_{r,n}$ belong to disjoint blue paths.

The situation is exactly the same for green paths.

Proof. Set up induction of the first assertion on the cardinal k of SDs. For $k = 1$ it is trivial. Suppose the result is true for $k = s$, and let a_{p_0,q_0} be the tour entry in this SD. We claim that

(1) the edge configurations from a_{p_0,q_0} to a_{q_0,p_0} must be the same, and
(2) the 2 blue edges at each entry are on different sides of the SD.

We divide into 2 cases, illustrated below by figure 3.

The first case is $a_{p_0,q_0} = 1$ or -1. If not all the blue edges of the entries are vertical or horizontal, then a pair of adjacent entries would be connected together, contradicting induction hypothesis.

The second case is $a_{p_0,q_0} = 0$, the corner shapes formed by the blue edges must agree. Otherwise, find a pair of adjacent entries, a_{p_1,q_1} and $a_{p_2,q_2}$with distinct corner shape in the SD. Then, the color of either their LHS edges or their bottom edges are distinct. WLOG, assume the former possibility. Now going backwards along the tour, we can find the first SD, say of cardinal $k - t$, with nonzero entries. Consider a_{p_1,q_1-t} and a_{p_2,q_2-t} on this SD. Both of these entries are in between the tour entry and its reflexive counterpart on $(k - t)$th SD. However, since all the entries in between a_{p_1,q_1}(included) and a_{p_1,q_1-t}(excluded) are 0, the LHS edge color alternates t times, and same for in between a_{p_2,q_2} and a_{p_2,q_2-t}. So the colors of LHS edges of a_{p_1,q_1-t} and a_{p_2,q_2-t} are distinct, contradicting the first situation.

Now if $q_0 > p_0$, then the 2 blue edges must be on different sides of the SD, otherwise the (at least 2) entries in between would be in a single blue path. In the case $q_0 = p_0$ and $a_{p_0,p_0} \neq 0$, if they are on the same side, then the LHS edge

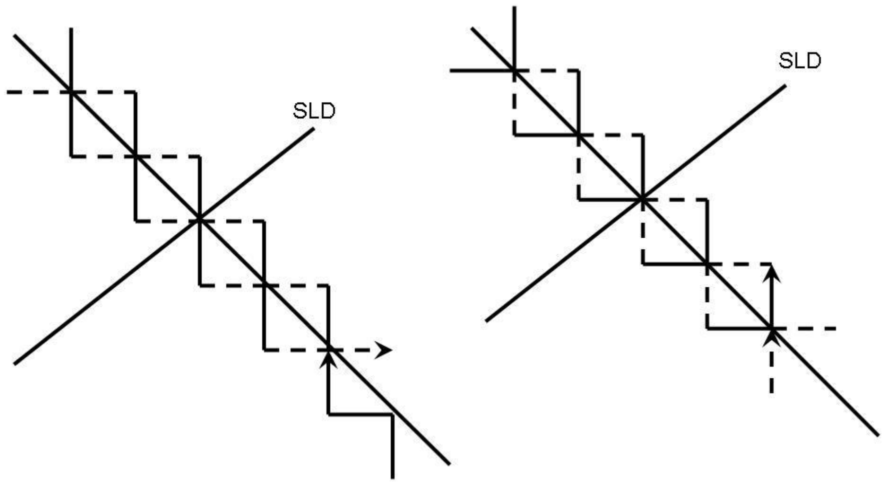


FIGURE 3. FPL configuration on kth skew diagonal, between a_{p_0,q_0} and a_{q_0,p_0}. Left: $a_{p_0,q_0} \neq 0$; Right: $a_{p_0,q_0} = 0$. The edges with arrows refers to tour edges.

of a_{p_0,p_0-1} and the RHS edge of a_{p_0-1,p_0} will be the same, which contradicts the switching of colors by the reflection.

Now let $k = s + 1$ and the next tour entry be $a_{p,q}$, which is 1 edge away from a_{p_0,q_0}. From $a_{p,q}$ to $a_{q,p}$, the LHS and bottom edges have the same color, by hypothesis. Since every entry is equal, their final edge configuration are the same. But since all corresponding edges are parallel, the original blue paths still keep disjoint. (1) is proved.

For (2), we set up induction hypothesis on cardinal k of skew diagonals, that all blue edges of entries (and external vertices) below the tour entry a_{p_0,q_0} on the kth SD are on the same side of kth SD, and so are all green edges. This time we start from $(2n-1)$th SD, which is trivial.

First note each entry below a_{p_0,q_0} on kth SD is 0. Assume the claim works for $k = r$. Note that if $k = r$ is even, then by simple odd/even parity argument, all blue edges of entries below a_{p_0,q_0} are below the rth SD. Consider the entry a_{p_0-1,q_0}, on $(r-1)$th SD. Then all entries on $(r-1)$th SD below a_{p_0-1,q_0} have the green edges below, and the blue edges above $(r-1)$th SD. Divide into 2 cases:

(1) a_{p_0-1,q_0} is a tour entry, then we are done.
(2) Otherwise, the tour entry on $(r-1)$th SD is a_{p_0,q_0-1}, its RHS tour edge is green, and the bottom edge of a_{p_0,q_0} is blue, by above proof of (1). Thus both green edges of a_{p_0-1,q_0} are below $(r-1)$th SD.

So the claim holds for $k = r - 1$, with the case $k = r$ odd being the same except reversing all the colors. Thus all blue edges of entries and external vertices below a_{p_0,q_0} on kth SD are in a single blue path (same for green). An illustration is shown in Figure 4.

□

Combining lemma 3.8 and proposition 3.9, we can easily see that the SDA is equivalent to forming the tour by the rules in lemma 3.8, setting the entries below the tour as 0, do SDF along the tour, and then do RC. From the above proof, we also see that closed loops of blue(green) edges never appear.

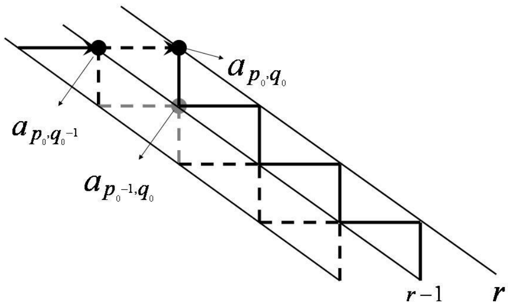


FIGURE 4. FPL configuration on a skew diagonal, for entries below the tour entry. The edges with arrows refer to tour edges.

Lemma 3.10. *Suppose a tour entry $a_{p,q}$, $q < n$, is on kth SD, and the previous tour entry is a_{p_0,q_0}. Then*

(1) *If the RHS edge of $a_{p,q}$ is blue, then the blue path γ_1 at a_{p_0,q_0} is joined to the blue path γ_2 below the $(k+1)$th SD.*
(2) *Otherwise, γ_1 and γ_2 never meet.*

Blue can be replaced by green.

Proof. (1): $a_{p_0,q_0} = a_{p-1,q}$ or $a_{p,q-1}$, so γ_1 is 1 horizontal blue edge away from γ_2. By connecting this blue edge, the two blue paths are joined.

(2): Otherwise, the RHS edge is green, and they are not joined. The next tour entry is $a_{p,q+1}$, and both γ_1 and γ_2 cross the $(k+1)$th SD above $a_{p,q+1}$ and below $a_{q+1,p}$. By proposition 3.9, they never meet. □

Adopting notations from lemma 3.10, we have

Proposition 3.11. *For tour entry $a_{p,q}$, $1 \le q < n$, $q \ge p$, suppose the previous tour edge is blue(green). Then the two external vertices, one in γ_1 and the other in γ_2, coincide with the blue(green) question checked at $a_{p,q}$. Thus by lemma 3.10 the link status of the questioned pair on the FPL diagram agrees with the answer. If $p = q+1$, then the RHS tour edge of $a_{p,p-1}$ ensures that at $a_{p,p}$, the two vertices in γ_1 and γ_2 are p and $2n+1-p$, coinciding with the question $?(p, 2n+1-p)$ checked at $a_{p,p}$.*

Proof. Do induction on the number k of skew diagonals. For $k = 1$, it is the same as $?(1, 2n)$. If it is false, the next tour entry is $a_{1,2}$ with the green question $?[1, 2]$. Otherwise, the next tour entry is $a_{2,1}$, and $a_{2,2}$ is joined by the blue edge to vertex 2. The blue path below the 4th SD is from blue vertex $2n-1$, so the next question is $?(2, 2n-1)$ by reflecting at $a_{2,2}$. Suppose the theorem is true for $k = l-2$ and $k = l-1$. We should prove it goes on to $k = l$ and $l+1$.

First assume both the tour entries at $(l-2)$th and $(l-1)$th SD are below or on the SLD. Let the tour entry at $(l-1)$th SD be a_{p_0,q_0} and the previous edge is blue. The two blue paths γ_1, γ_2 come from blue vertices r_0,s_0 respectively. By hypothesis, the blue question checked at a_{p_0,q_0} is $?(r_0, s_0)$.

If $?(r_0, s_0)$ is false, then the next tour entry is a_{p_0,q_0+1}. Consider the blue path γ_3 from blue vertex $s_0 - 1$, which is below $(l+2)$th SD by proposition 3.9. At a_{p_0,q_0+1}, γ_2 and γ_3 are 2 horizontal edges apart. But regardless of the answer at a_{p_0,q_0+1}, the next tour edge is blue, and γ_2 comes 1 horizontal edge closer to γ_3.

(False answer is obvious, and true answer is demonstrated in Figure 5.) By lemma 3.10, the two vertices concerned at the tour entry a_{p_1,q_1} on $(l+1)$th SD coincide with the blue question $?(s_0, s_0-1)$ checked at a_{p_1,q_1}.

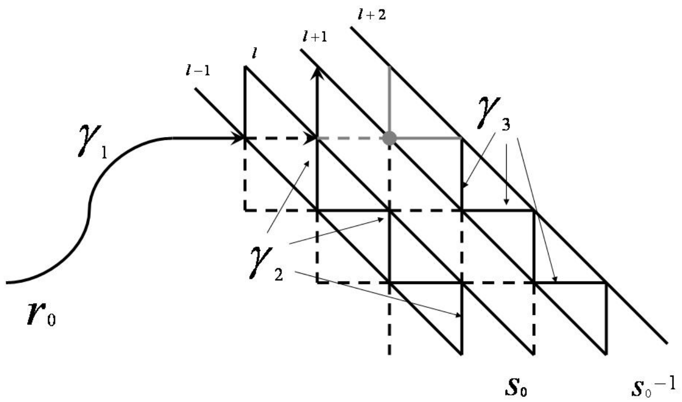


FIGURE 5. The induction from $(l{-}1)$th SD to $(l{+}1)$th SD, $?(r_0, s_0)$ false. $a_{p_0,q_0} = 0$, the question checked at a_{p_0,q_0+1} is true, i.e. $a_{p_0,q_0+1} = 1$ and the shaded edges (with the dot) are thus determined.

Now let $?(r_0, s_0)$ be true. Then the next blue question would be at a_{p_1,q_1} on $(l+1)$th SD. Let γ_1 be the blue path at a_{p_1,q_1}, and γ_2 the blue path just below $(l+2)$th SD. Same as above, γ_2 comes from $s_0 - 1$. Suppose γ_1 joins the blue vertex r_1. We claim that r_1 is the first blue vertex clockwise from r_0 that belongs to no links confirmed by a question checked at any preceding tour entry . Then it coincides with the blue question checked at a_{p_1,q_1}. In both cases, the link status agrees with the answer as lemma 3.10 indicates.

To prove our claim, we see that r_1 cannot be any of the confirmed vertices, by part 2 of lemma 3.10. Now as we go along $(l+1)$th SD from a_{p_1,q_1} (positively), the blue paths passing through each entry are below lth SD and disjoint. These paths are all bounded below by γ_1, which joins r_1. This means r_1 is the first unconfirmed vertex clockwise from r_0, as desired. This is illustrated in Figure 6.

The induction from $l-2$ to l of green questions is analogous, except the entry on lth SD is above the SLD and the blue path passing through this entry joins a_{p_1,q_1}, and by above induction from $l-1$ to $l+1$ the blue path actually links vertex p_1.

Then assume the tour entry on $(l-2)$th SD is above the SLD. This entry is $a_{p,p-1}$. By induction hypothesis, the two desired vertices at $a_{p,p}$ on $(l-1)$th SD are p and $2n+1-p$. Actually, $l = 2p$. If the answer is true, the next tour entry is $a_{p+1,p}$. The blue path γ_1 passing through $a_{p+1,p}$ connects to vertex $p+1$, since the reflection image of γ_1, which is a green path, passes through $a_{p,p+1}$ and by proposition 3.9 goes down to the green vertex $p+1$. By the RHS blue edge γ_1 connects $a_{p+1,p+1}$, and γ_2 below $(l+2)$th joins $2n-p$. Otherwise, the next tour entry is $a_{p,p+1}$. The color question to be checked is then $?[p, p+1]$. But the RHS edge of $a_{p,p}$ is green, meaning that $a_{p,p+1}$ is connected to green vertex p by a green path. The green path below $(l+1)$th SD links $p+1$, so we are done.

□

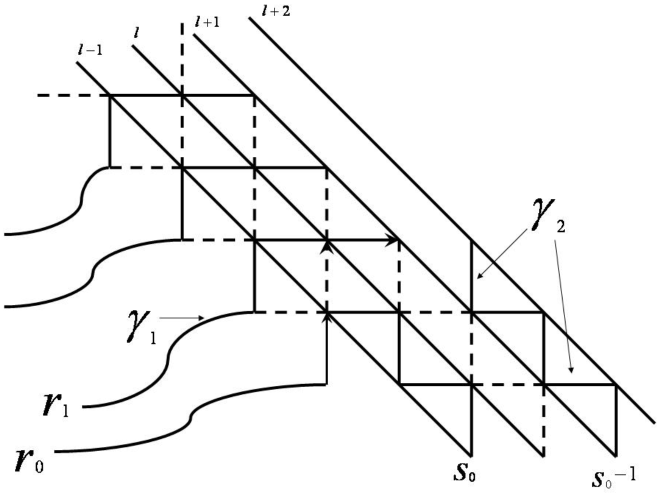


FIGURE 6. The induction from $(l-1)$th SD to $(l+1)$th SD, $?(r_0, s_0)$ true

Thus, before the tour reaches nth column, the ASM fits perfectly with all the previous answers, which are independent in the sense of proposition 3.4. The last step is to show the remaining construction of the ASM is in accordance with the subsequent true answers.

Proposition 3.12. *If $a_{p,n} = 1$, then the $n-p$ consulted but unconfirmed vertices, labelled clockwise as $t_1, ...t_{n-p}$, are linked on the lattice to the vertices $t_1 - 1, ..., t_1 - n + p$ respectively. This is consistent with π_0 and thus $\pi_B = \pi_0 = \pi_G$.*

Proof. Just before the tour arrives at $a_{p,n}$, by proposition 3.11 the link status of all consulted vertices on the FPL agrees with π_0. By proposition 3.4, these questions and answers uniquely determine π_0 and yield the subsequent blue or green links, $(t_1, t_1 - 1), ..., (t_{n-p}, t_1 - n + p)$. By setting $a_{r,n} = 1$, we confirm the link $(t_1, t_1 - 1)$. On $(n + p - 1)$th SD, all entries are 1, so by proposition 3.9, all entries belong to disjoint blue paths, so the $(n - p + 1)$ links must be arranged clockwise from t_1 to t_{n-p}. Now proposition 3.11 continues to hold, except that γ_2 from $t_1 - i$ consists of a single blue or green edge from the vertex and is above the SD. $a_{r,n}$=0 if $r > p$ since we always fill in vertical edges alternating in blue and green. This corresponds to confirming a trivially true question.

□

Thus, theorem 3.1 is proved.

4. ASMs with Antipodal Blue and Green Link Patterns

Recall π_1 is antipodal to π_0 if π_1 differs from π_0 by the 180 degree rotation. By reflecting a FPL diagram about the matrix diagonal, and switching the colors of all edges, we switch blue pairings (r, s) to the antipodal green ones $[r + n, s + n]$ (mod $2n$). In particular, if the ASM is invariant under reflection by the matrix diagonal, then π_B and π_G are antipodal to each other. Therefore our algorithm to construct ASMs with antipodal blue and green link patterns is essentially a vertical reflection version of the SDA. But to make things precise, we need some definitions.

Definition 4.1. A *lattice diagonal(LD)* is a line of matrix entries parallel to the standard lattice diagonal in definition 2.8. The *kth lattice diagonal*, $k = 1, ..., 2n-1$, is the set of entries:

- $k = n$: $a_{r,r}, r = 1, ..., n$ (the standard lattice diagonal)
- $k < n$: $a_{r,n-k+r+1}, r = 1, ..., k$
- $k > n$: $a_{k-n+1+r,r}, r = 1, ...2n - k$

The *positive direction* of a lattice diagonal is going from the bottom left entry to the top right entry.

Definition 4.2. A *pseudo-tour* in an FPL state is a path of edges such that

(1) it starts from the external vertex opposite to the blue vertex 1;
(2) it only goes leftwards and upwards;
(3) blue and green edges in the path alternate.

A pseudo-tour entry(pseudo-tour edge) is an entry(edge) in the pseudo-tour. The pseudo-tour can also be realized in the corresponding ASM, where the starting entry is the bottom right entry.

Now for a link pattern π_0, let π_1 be its antipodal link pattern. From now on, let $\beta_0 = \lfloor \frac{n}{2} \rfloor + 1$ and $\alpha_0 = \beta_0 + n$. Any number r referring to a vertex is realized as $r \pmod{2n}$.

Definition 4.3. We define a sequence $\overline{B_\pi}$ of *pseudo-blue questions*, $?(r_k, s_k)$, and a sequence $\overline{G_\pi}$ of *pseudo-green questions*, $?[u_k, v_k]$. Both link patterns are *on the circle.* Each pseudo-color question asks for whether two vertices i, j are in a link *in* π_0, and belongs to only one color. Similar to color questions, we define pseudo-color questions within one cancellation cycle.

First, let n be odd. Then all questions $?(\alpha_0 + 1 - p, \alpha_0 + p)$ are pseudo-blue questions. Otherwise n is even, and $?[\alpha_0 - 1 + p, \alpha_0 - p]$ are pseudo-green questions; in both cases, $p = 1, ..., n$. Every cancellation cycle starts from a pseudo-color question of these types and ends before the next such question arise. $\overline{B_\pi}$ and $\overline{B_\pi}$ combine to form the pseudo-color question sequence $\overline{Q_\pi}$ by the order of appearance of each question.

So now we fix a cancellation cycle. In the cycle, the first pseudo-blue question is $?(\alpha_0 + 1 - q, \alpha_0 + q)$. Suppose $?(r_k, s_k)$ is asked, where s_k is between the circular arc *clockwise* from $\alpha_0 + 1$ to β_0 for n odd and to $\beta_0 - 1$ for n is even. Then the next pseudo-blue question (if applicable) is $?(r_{k+1}, s_{k+1})$ where $s_{k+1} = s_k + 1$, and

- $r_{k+1} = s_k$ if $?(r_k, s_k)$ is false;
- r_{k+1} is the first blue vertex counterclockwise from r_k, such that r_{k+1} is NOT confirmed by any previous pseudo-color question (not restricted in this cycle);

The first pseudo-green question in the cycle is $?[\alpha_0 - 1 + q, \alpha_0 - q]$. If $?[u_k, v_k]$ is the kth pseudo-green question, where v_k is between the arc counterclockwise from $\alpha_0 - 1$ to β_0 for n even and to $\beta_0 + 1$ for n odd. Then the next pseudo-green question (if applicable) is $?[u_{k+1}, v_{k+1}]$ where $v_{k+1} = v_k - 1$, and

- $u_{k+1} = v_k$ if $?[u_k, v_k]$ is false;
- u_{k+1} is the first green vertex clockwise from u_k, such that u_{k+1} is NOT confirmed by any previous pseudo-color question.

If n is odd, the last possible pseudo-green question is when $v_k = \beta_0 + 1$ and the last possible pseudo-blue question is when $s_k = \beta_0$. If n is even, the last possible pseudo-green question is when $v_k = \beta_0$ and the last possible pseudo-blue question is when $s_k = \beta_0 - 1$.

It is easy to see that pseudo-color questions are blue and green questions with different initial condition, with pseudo-blue(green) questions corresponding to green(blue) questions. Therefore similar to color questions, no pseudo-color question repeats itself.

Similar to the two constructions SDF and RC, we have

Definition 4.4. In an incomplete matrix, if an entry $a_{r,s}$ is determined, then the *Lattice Diagonal Filling(LDF)* process is to set $a_{p,q} = a_{r,s}$ if

(1) $a_{p,q}$ is on the LD of $a_{r,s}$, and;
(2) $a_{p,q}$ is above $a_{r,s}$ and below or on the matrix diagonal, i.e. $p > r$ and $p + q \leq n + 1$.

And the *Lattice Reflection Construction(LRC)* is to set $a_{p,q} = a_{r,s}$ where $a_{p,q}$ is the image of $a_{r,s}$ under reflection about the matrix diagonal, i.e. $p = n+1-s, q = n+1-r$.

Fixing a link pattern π_0 of $2n$ vertices, we are now in a position to define the *Lattice Diagonal Algorithm(LDA)*. However, writing the algorithm in the form of section 2.2 is clumsy and obscures the great similarities between these two algorithms. Recall lemma 3.8, which provides a simple tic-tac-toe approach to define SDA. We would like to include the LDA in the following theorem.

Theorem 4.5. *Let π_1 be the antipodal link pattern of π_0. We construct an $n \times n$ FPL state as follows. Before we start to work on an empty square ice lattice, record the sequence of pseudo-color questions $\overline{Q_\pi}$ and the answers.*

First, construct a pseudo-tour in the square ice lattice. The pseudo-tour starts from $a_{1,n}$, and check the answer of first pseudo-blue question $?(\alpha_0, \alpha_0 + 1)$ for n odd; the first pseudo-green question $?(\alpha_0, \alpha_0 - 1)$ for n even. In general, at an undetermined pseudo-tour entry, we check the kth pseudo-color question in $\overline{Q_\pi}$ if it is the kth pseudo-tour entry below or on the matrix diagonal, and fill the next pseudo-tour edge by the following rules:

(1) *Blue and green colors alternate;*
(2) *If a top edge appears on top of $a_{r,n+1-r}$, then the next tour edge is on the LHS of $a_{r+1,n+1-r}$ (actually for n odd this edge is blue, for n even it is green);*
(3) *If the question checked at $a_{p,q}$ is true, then fill the next edge on top of the entry; otherwise, fill it on the LHS.*

After the pseudo-tour in the square ice lattice is completed, all pseudo-tour entries in the corresponding ASM are determined. Then do LDF along each pseudo-tour entry, set all the entries below the pseudo-tour as 0, and then do LRC for all entries.

Then the matrix so constructed is an ASM with $\pi_B = \pi_0$ and $\pi_G = \pi_1$.

The proof of theorem 4.5 imitates theorem 3.1, and most results can actually be adopted by making analogues in this situation. First, determination of π_0 and thus π_1 by pseudo-color questions are exactly the same, since pseudo-color questions are basically color questions with different initial conditions. Second, the observations in section 3.2 applies here, except to replace tour by pseudo-tour, questions by pseudo-color questions, the skew diagonals by lattice diagonals(LD), the identical blue and green pairings by antipodal ones, etc. We will not present the detailed proof of these analogue propositions. Still, we need some counting lemma like lemma 3.5 to prove that pseudo-tour always reach the top left corner $a_{n,1}$ and thus LDA completes a matrix. We state the lemma as follows:

Lemma 4.6. *When the pseudo-tour proceeds to $a_{p,q}$, there are $n+2-p-q$ vertices if $p+q < n+1$, or if $p+q = n+1$ and the previous pseudo-tour entry is $a_{p-1,n+1-p}$. Otherwise, there is none at the pseudo-tour entry $a_{p,n+1-p}$.*

We reorganize the flow of proof in section 3 and outline the proof of theorem 4.5.

Proof of theorem 4.5. First, the rules of pseudo-tour implies the filling of pseudo-tour entries in a matrix in the same way as section 2.2, except to replace rightward movement by leftward, and the next two pseudo-tour entries are $a_{p+1,n+1-p}$ and $a_{p+1,n-p}$ if the answer of the pseudo-color question at $a_{p,n+1-p}$ is true. By analogue of proposition 3.4, the counting lemma 4.6, and construction of LDA, the pseudo-tour will have a first entry $a_{p,1}$ in the leftmost column and then arrive at $a_{n,1}$, so the matrix is complete. By similar entry chasing as in proposition 3.7, the matrix is an ASM with π_B and π_G antipodal to each other.

Similar to proposition 3.9 and 3.10, we find that, on the lattice diagonal of a pseudo-tour entry $a_{p,q}$, entries below belong to a single blue(green) path and entries between $a_{p,q}$ and $a_{n+1-q,n+1-p}$ belong to disjoint paths, and a choice of LHS edge of the pseudo-tour entry directly confirms or falsifies the link between two vertices.

We set up induction on the cardinal l of lattice diagonals (from $l-2, l-1$ to $l, l+1$) of pseudo-tour entries, and claim that, as long as $q > 1$,

(1) If the question checked at $a_{p,q}$ is pseudo-blue $?(r,s)$, r, s are the two vertices from γ_1 at $a_{p,q}$ and γ_2 below the $(l+1)$th lattice diagonal;
(2) If it is pseudo-green $?[u,v]$, then the two vertices above are $u+n$ and $v+n$, i.e. antipodal to u, v.
(3) If the pseudo-entry is above the matrix diagonal, then its LHS edge ensures that the vertices in γ_1 and γ_2 are α_0+1-p and α_0+p for n odd; α_0-1+p and $\alpha_0-p)$ for n even.

Let a_{p_0,q_0}, a_{p_1,q_1} be the pseudo-tour entries on $(l-2)$th, $(l-1)$th LD respectively. At most one of them is above the matrix diagonal. We can assume the pseudo-tour entry a_{p_1,q_1} on $(l-1)$th LD is below or on the matrix diagonal, since this already contains the case when a_{p_0,q_0} is so. Let the question at a_{p_1,q_1} be $?(r_0,s_0)$ if pseudo-blue, $?[u_0,v_0]$ if pseudo-green.

First let the pseudo-tour entry a_{p_0,q_0} on$(l-2)$th LD be below or on the matrix diagonal also. The color of question at a_{p_2,q_2} on $(l+1)$th LD is the same as that at a_{p_1,q_1} (on $(l-1)$th LD). We verify the induction from $l-1$ to $l+1$ by noting that the blue(green) path γ_1 at the pseudo-tour entry on $(l+1)$th LD is the lowest one among all disjoint paths above this entry. If the question is $?(r_0,s_0)$ at a_{p_1,q_1}, then γ_1 connects to the first unconfirmed vertex counterclockwise from s_0+1 (in the blue path below $(l+2)$th LD), so it coincides with the pseudo-blue question $?(r_1,s_0+1)$ checked at a_{p_2,q_2}. If the question is $?[u_0,v_0]$ at a_{p_1,q_1}, then γ_1 connects to the first unconfirmed vertex counterclockwise from v_0-1+n, which is then u_1+n. So it coincides with the pseudo-green question $?[u_1,v_0-1]$ checked at a_{p_2,q_2}.

Then let a_{p_0,q_0} on $(l-2)$th LD be above the diagonal. We have to separate n being odd and even. Let n be odd. The question at a_{p_0,q_0-1} is by hypothesis $?(\alpha_0+1-p_0, \alpha_0+p_0)$. If the answer is true, the blue vertex α_0-p_0 is connected to a_{p_0+1,q_0-1} and then by the LHS blue edge to a_{p_0+1,q_0-2}. This matches the question checked at a_{p_0+1,q_0-2} which is $?(\alpha_0-p_0, \alpha_0+p_0+1)$. If the answer is false, a cancellation cycle starts and the next question is pseudo-green $?[\alpha_0+1-p_0, \alpha_0-p_0]$, which is proved in the above paragraph.

Let n be even. The pseudo-green question at a_{p_0,q_0-1} is by hypothesis $?[\alpha_0-1+p_0, \alpha_0-p_0]$. If the answer is true, then the green vertex $\beta_0+p_0=\alpha_0+p_0-n$ is connected to a_{p_0+1,q_0-1} and then by the LHS green edge to a_{p_0+1,q_0-2}, and this matches the next question $?[\alpha_0+p_0, \alpha_0-p_0-1]$ checked at a_{p_0+1,q_0-2}. The false answer is same as above.

Finally, the situation after the first column is reached by the pseudo-tour is again analogous to proposition 3.12.

□

5. Application of the Results

Given a link pattern π_0, we already have SDA and LDA in our disposal. If π_0 is *not* antipodal to itself, then obviously these two algorithms generate distinct ASMs. However, this is far from imposing strong enough restrictions on π_0 to be achieved by a unique ASM.

A wonderful tool to simplify our investigation is a weak form of the Wieland's theorem [4]:

Theorem 5.1. *For link patterns π_0 and π_1, if π_1 is obtained from π_0 by a dihedral action, i.e. composition of a rotation of $k\pi/n$ with a reflection, then the set of $ASM(n)$ with $\pi_B = \pi_0$ is in bijection to the set of $ASM(n)$ with $\pi_B = \pi_1$.*

We recall and adopt the notation in 3.3. It is trivial to see each link pattern has at least 2 adjacent links. On the other hand, the link pattern $\pi^0 = (r_1, s_1), ..., (r_n, s_n)$ has exactly 2 adjacent links, and indeed, up to dihedral symmetry, this is the unique pattern that has such a property:

Lemma 5.2. *A link pattern π_1 has exactly 2 adjacent links if and only if π_1 and π^0 differ by the action of a suitable rotation composed with a reflection.*

Proof. Suppose π_1 has exactly 2 adjacent links. By a dihedral action, we may assume one of the adjacent link to be $(1, 2n)$. This is exactly the first color question defined in definition 2.2. We claim that all subsequent $n-1$ color questions are true, and this directly implies the lemma.

From the first question $(1, 2n)$, let the first false answer appear at the pth question. Here $p < n$. Then the next true answer, which exists by virtue of proposition 3.4, confirms an adjacent link. Now if this is the point of determination of π_1, then by proposition 3.4, there is another adjacent link, namely, $(t_1 - n + p, t_1 - n + p + 1)$, distinct from the previous 2. This is impossible. Otherwise, continue asking questions up to the determination point, and same as above, we will find another adjacent link distinct from the previous 2.

□

Definition 5.3. A link is called *strictly bottom adjacent link* if it is a bottom adjacent link and both vertices belong to the set $r_1, ..., r_n$ defined in definition 3.2.

Lemma 5.4. *Suppose π_1 has more than 2 adjacent links, and let π_2 be the antipodal link pattern of π_1 (they can be the same). Then either π_1 or π_2 has 2 bottom adjacent links and one of them is strictly bottom.*

Proof. Let $T = \{s_1, ..., s_n\}$ be the set complement to the set of bottom vertices $B = \{r_1, ..., r_n\}$. T is indeed the set antipodal to B. If π_1 has at least 3 adjacent links, then there exists 2 adjacent links with vertices in either T or B, and one of them has both vertices in the set chosen above. If this set is T, then we choose π_2, otherwise we choose π_1.

□

Proposition 5.5. *If π_1 has more than 2 adjacent links, then it is achieved by at least 2 distinct ASMs.*

Proof. By theorem 5.1, the number of ASMs with $\pi_B = \pi_1$ is equal to that with $\pi_B = \pi_2$, and by lemma 5.4, we may, WLOG, consider π_1 having a strictly bottom adjacent link. The SDA generates a 1 at the closest adjacent link to blue vertex 1, and all other bottom row entries 0. But the LDA generates a 1 at the *farthest* adjacent link from blue vertex 1. The location of the corresponding entries for these two adjacent links are distinct, so the resultant ASMs are distinct.

□

Thus the necessary part of theorem 1.5 is established.

Remark 1. The proof of proposition 5.5 does not contradict the case of exactly 2 adjacent links, since either there is only 1 strictly bottom adjacent link, or the 2 non-strictly bottom adjacent links coincide at the corner. Thus SDA and LDA do not produce distinct ASMs.

Now we move on to prove the sufficiency part of theorem 1.5.

First we need the definition of *gyration* operation, which is due to Wieland [4].

Definition 5.6. Gyration is an operation G: $\mathrm{ASM}(n) \mapsto \mathrm{ASM}(n)$ defined as follows.

First we label the unit squares created in the square ice lattice. Label each vertex (i, j) [1], where the bottom left vertex is $(1, 1)$ and numbers increase as you move up and right. A unit square is labeled by parity of $i + j$, where (i, j) is the box's bottom left vertex. So now the unit squares are divided into even and odd boxes.

First define G_{even}. Visit every even box. If one pair of opposite edges in the box is blue and the other pair is green, then switch the colors of all edges. Otherwise, the box remains the same. G_{odd} is defined in the same way, except by acting on the odd unit squares. Then the gyration is defined as $G = G_{even} \circ G_{odd}$.

Lemma 5.7. *Let $A \in ASM(n)$ such that it is the only ASM with its specific blue link pattern π_0. Then the blue link pattern of $A' = G(A)$ can be achieved only by A'.*

Proof. Let the blue link pattern of A' be π_1. Suppose that π_1 can be achieved by B distinct from A'. But that means that both $G^{-1}(A')$ and $G^{-1}(B)$ must have the same blue link pattern, namely π_0. But since π_0 can only be achieved by A, $G^{-1}(A') = G^{-1}(B) = A$. This is contradiction, since G is a bijection. So π_1 is achieved only by A'.

□

Now we are ready to prove the following theorem.

Theorem 5.8. *There exist at least n blue link patterns such that it is achieved by a unique ASM(n).*

Proof. Note that in any FPL of size n, there are exactly $n(n-1)$ blue edges. Using this fact, we will prove Theorem 5.8.

Lemma 5.9. *The fewest number of blue edges necessary to achieve a blue pairing with two adjacent links can be realized in the following way. First create the two adjacent links in the simplest way. After creating link (i, j) with $i < j$, link $(i - 1, j + 1)$ in the shortest way that fits.*

Proof. Assume that pair (i, j) is not drawn in the shortest way possible, and that all links nested within it are. That means that this blue edge must extend further into the middle of the diagram. But that means that, in order to avoid two blue edges crossing, (i-1,j+1) must also bulge outward and no longer take it's shortest path. This will continue until you reach the other adjacent link, which is already as short as possible. It cannot become any shorter because of some other edge being longer. Thus, this is not the smallest possible blue edges. So the greedy algorithm does produce the fewest blue edge segments. □

We first find one ASM with a unique matching in four distinct cases.

Case 1. $n \equiv 0$ *mod* 4.

[1] Please note this is not to be confused with the link notation.

Start with a blank square ice lattice and draw the shortest path linking $(\frac{n}{4}, \frac{n}{4}+1)$. After linking (i, j) with $j > i$, link $(i-1, j+1)$ with shortest possible path. The first link has 2 blue edges, and the next link must have 8, etc. In general, it is easy to check that the ith link for $i <= n/4$ (the left side) will have $(6i-4)$, making the total for the left side $\sum_{i=1}^{n/4}(6i-4) = 3\left(\frac{n}{4}\right)^2 - \frac{n}{4}$. By symmetry, the sum will be the same on the left.

Reindex the links so that links 1 through $\frac{n}{4}$ are the leftmost links between the top and bottom of the FPL. Then it is easy to see that the ith link will have $(\frac{n}{4} - i + 1) + (\frac{n}{4} - i) + (n-1)$ edges, making the total contribution from top-to-bottom links

$$2\sum_{i=1}^{n/4}\left[(\frac{n}{4} - i + 1) + (\frac{n}{4} - i) + (n-1)\right] = 2\left[\left(\frac{n}{4}\right)^2 + \frac{n}{4}(n-1)\right].$$

Then the total number of blue edges is

$$2\left[\left(\tfrac{n}{4}\right)^2 + \tfrac{n}{4}(n-1)\right] + 2\left[3\left(\tfrac{n}{4}\right)^2 - \tfrac{n}{4}\right] = n(n-1).$$

But this is the maximum blue edges allowed, and since the shortest link paths are unique, there can only be one ASM with this link pattern.

Case 2. $n \equiv 2 \ mod\ 4$.

Start with a blank FPL and draw the shortest path linking $(\frac{n+2}{4}, \frac{n+2}{4}+1)$. Continue adding loops as in Case 1. It is easy to check that the ith link for $i < n/4$ (the left side) will have $(6i-4)$ edges and the $(\frac{n+2}{4})$th loop has $6\left(\frac{n+2}{4}\right) - 5$, making the total for the left side

$$\left[\sum_{i=1}^{(n-2)/4}(6i-4)\right] + 6\left(\frac{n+2}{4}\right) - 5 = 3\left(\frac{n+2}{4}\right)^2 - \frac{n+2}{4} - 1.$$

By symmetry, this number will be the same for the right side.

Reindex the links so that links 1 through $\frac{n-2}{4}$ are the leftmost links between the top and bottom of the FPL. Then it is easy to see that the ith link will have $(\frac{n+2}{4} - i - 1) + (\frac{n+2}{4} - i) + (n-1)$ edges, making the total top-to-bottom contribution

$$2\sum_{i=1}^{\frac{n-2}{4}}\left[(\frac{n+2}{4} - i - 1) + (\frac{n+2}{4} - i) + (n-1)\right] = 2\left[\left(\frac{n+2}{4}\right)^2 + \frac{n+2}{4}(n-5) - n + 2\right]$$

Adding them all together,

$$2\left[\left(\tfrac{n+2}{4}\right)^2 + \tfrac{n+2}{4}(n-5) - n + 2\right] + 2\left[3\left(\tfrac{n+2}{4}\right)^2 - \tfrac{n+2}{4} - 1\right] = n(n-1).$$

Again, this is the maximum number of blue edges allowed, and since the shortest link paths are unique, there can only be one ASM with this link pattern.

Case 3. $n \equiv 1 \ mod\ 4$.

Start with the blank square ice lattice and draw the shortest path to link $(\frac{3n+1}{4}, \frac{3n+1}{4}+1)$. Continue adding links as in Cases 1 and 2. It is easy to check that the ith link for $i < (n-1)/4$ (the top side) will have $(6i-4)$ edges, making the total contribution $\sum_{i=1}^{(n-1)/4}(6i-4) = 3\left(\frac{n-1}{4}\right)^2 - \frac{n-1}{4}$.

By symmetry, this number will be the same for the bottom.

Reindex the links so that links 1 through $\frac{n-1}{4}$ are the highest links between the left and right of the FPL. Then it is easy to see that the ith link will have $(\frac{n-1}{4} - i + 1) + (\frac{n-1}{4} - i + 1) + (n-1)$ edges and the middle link will have $n-1$ edges, making the total contribution

$$2\left[\sum_{i=1}^{\frac{n-1}{4}} (\frac{n-1}{4} - i + 1) + (\frac{n-1}{4} - i + 1) + (n-1)\right] + (n-1)$$

$$= 2\left[\left(\frac{n-1}{4}\right)^2 + \frac{n-1}{4}(n)\right] + n - 1.$$

This again sums to $n(n-1)$, and by the same argument, the described link pattern gives a unique ASM.

Case 4. $n \equiv 3 \ mod\ 4$ and $n > 3$ [1]

Start with the blank square ice lattice and draw the shortest path to link $(\frac{n+1}{4}, \frac{n+1}{4} + 1)$. Continue adding links as in the previous cases. It is easy to check that the ith link for $i \leq (n+1)/4$ (the left side) will have $6i - 4$ lines, making the total contribution

$$\sum_{i=1}^{(n+1)/4} (6i - 4) = 3\left(\frac{n+1}{4}\right)^2 - \frac{n+1}{4}$$

By symmetry this number will be the same for the right side.

Reindex the links so that links 1 through $\frac{n-3}{4}$ are the leftmost links running from top to bottom. Then it is easy to see that the ith link will have $(\frac{n-3}{4} - i + 1) + (\frac{n-3}{4} - i + 1)$ and the center link will have $n-1$, making the total contribution

$$2\left[\sum_{i=1}^{\frac{n-3}{4}} (\frac{n-3}{4} - i + 1) + (\frac{n-3}{4} - i + 1) + (n-1)\right] + (n-1)$$

$$= 2\left[\left(\frac{n-3}{4}\right)^2 + \frac{n-3}{4}(n)\right] + n - 1$$

Summing these, we again find that the shortest paths take a total of $n(n-1)$ edges. So by the argument used earlier, the ASM with this link pattern is unique.

Note that in all of our examples, the link pattern is invariant under 180° rotation (i.e. self-antipodal), but not invariant under any smaller rotation. G^n defines a 180° rotation of the pairing, so $A, G(A), ...G^{n-1}(A)$ are n distinct matrices. By Lemma 5.7, the link patterns of these matrices can be achieved only by the corresponding matrices.

□

6. Possible Directions

Given a link pattern π_1, it is a natural question to ask for which link pattern π_2 does there exist an ASM(n) with $\pi_B = \pi_1$ and $\pi_G = \pi_2$. We are far from starting to solve this, but inspired by the theorem 1.2 and 1.3, we conjecture that

[1] The theorem can be checked for $n = 3$, but the proof fails.

Conjecture 1. *For two link patterns π_0 and π_1, if there exists an ASM with $\pi_B = \pi_0$ and $\pi_G = \pi_1$, then there also exists one with $\pi_G = \pi_2$ and π_2 is antipodal to π_1.*

Note that Wieland's result in [4] does not apply, since π_B is not rotated accordingly. We further conjecture, based on empirical evidence, that

Conjecture 2. *The set of ASM(n) such that $(\pi_B, \pi_G) = (\pi_1, \pi_2)$ is bijective to the set of ASM(n) with (π_1, π_3), where π_3 is antipodal to π_2.*

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